Journal of Applied Mechanics and Technical Physics, Vol. 36, No. 5, 1995

ASYMPTOTIC AVERAGING TECHNIQUE FOR HEAT CONDUCTION PROBLEMS WITH PHASE TRANSITIONS IN LAYERED MEDIA

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UDC 519.6

In describing many physical phenomena a need arises to consider the problems of heat conduction in materials undergoing phase transitions with heat release or absorption. An essential feature of these problems is the presence of a moving interface (front) between different phases. The work of Stefan devoted to the study of the thickness of polar ices is most likely the first dealing with analogous problems. A more general approach was formulated by Neumann (see [1]).

Description of propagation of a melting front is a sufficiently complicated linear problem whose exact solutions exist only for several particular cases of propagation of the melting (solidification) front in homogeneous bodies [2, 3]. The presence of inhomogeneities in the medium complicates considerably the solution of the heat conduction problems. Even in the absence of phase transitions in the nonhomogeneous medium there are only approximating techniques [4, 5], and the problem of propagation of a plane melting front in a layered medium is difficult even for numerical methods [6]. The motion of the interface, the law of motion being defined from the solution of the problem, prevents the direct use of the well-elaborated numerical methods (the finite-element method, the boundary-element method), which are efficient for media with fixed boundaries.

In the present work we propose to use the method of asymptotic averaging [7–10] used for nonhomogeneous media with periodical structure for solving the problem of heat conduction with allowance for the phase transitions. The object of the work is to obtain a relatively simple analytical expression that estimates of the dynamics of melting (solidification) processes in materials whose structure is close to periodical.

1. Let us consider a semi-infinite medium, which is a series of periodically alternating layers of different thickness (Fig. 1). The layers are arranged in parallel to a free surface, and the size of the periodicity cell equals H, i.e., for all parameters of the media the condition $R(x) = R(x + mH)$, $m = 1, 2, 3\ldots$ holds. For the given geometry we solve the problem of heat propagation inside the media, if at the moment $t = 0$ the temperature at the free surface becomes equal to T_1 and is maintained constant at $t > 0$. It is assumed that the temperature T_1 is higher than the initial medium temperature T_0. The temperature distribution in the medium will be governed by the nonstationary heat conduction equation

$$\rho(x)c(x)\frac{\partial T(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(k(x)\frac{\partial T(x,t)}{\partial x}\right) \tag{1.1}$$

with the initial condition

$$T(x,0) = T_0 \quad \text{at} \quad x > 0. \tag{1.2}$$

Here $T(x,t)$ is the temperature distribution; ρ, c, and k are the density, heat capacity, and heat conductivity of the medium.

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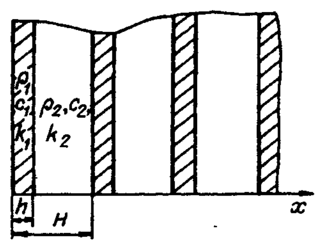

Fig. 1

In the case where $T_1 < \min\{T_1^*, \ldots, T_m^*\}$ (T_i^* is the melting point of the ith layer, $i = 1, \ldots, m$, m is the number of layers in the periodicity cell), Eq. (1.1) is supplemented by the boundary conditions

$$T(0,t) = T_1 \quad \text{at} \quad t \geqslant 0; \tag{1.3}$$

$$[T(x,t)]\Big|_G = 0; \tag{1.4}$$

$$\left[k(x)\,\frac{\partial T(x,t)}{\partial x}\right]\Bigg|_G = 0, \tag{1.5}$$

where square brackets denote a jump at the interface boundary G, i.e., $[T(x,t)]\,|_G = T(x,t)\,|_{G+0} - T(x,t)\,|_{G-0}$.

Within the framework of the asymptotic averaging technique [7] the solution of the problem (1.1)–(1.5) for a periodical medium is sought in the form

$$T(x,\xi,t) = T^{(0)}(x,\xi,t) + \varepsilon T^{(1)}(x,\xi,t) + \ldots + \varepsilon^n T^{(n)}(x,\xi,t). \tag{1.6}$$

Here ε is a small parameter equal to the ratio of the size of a periodicity cell H to the characteristic size of the problem ($\varepsilon \ll 1$); $\xi = x/\varepsilon$ is a "fast" variable which varies from 0 to H within each cell.

Let us substitute the expansion (1.6) into initial equation (1.1), having first made the passage to the variables (x, ξ, t). Then taking account of the differentiation rule of a composite function and equating the coefficients at equal powers of ε, we obtain

$$\frac{\partial}{\partial \xi}\left(k(\xi)\,\frac{\partial T^{(0)}}{\partial \xi}\right) = 0; \tag{1.7}$$

$$\frac{\partial}{\partial \xi}\left(k(\xi)\,\frac{\partial T^{(1)}}{\partial \xi}\right) + \frac{\partial}{\partial \xi}\left(k(\xi)\frac{\partial T^{(0)}}{\partial x}\right) + \frac{\partial}{\partial x}\left(k(\xi)\,\frac{\partial T^{(0)}}{\partial \xi}\right) = 0; \tag{1.8}$$

$$\frac{\partial}{\partial \xi}\left(k(\xi)\frac{\partial T^{(2)}}{\partial \xi}\right) + \frac{\partial}{\partial \xi}\left(k(\xi)\frac{\partial T^{(1)}}{\partial x}\right) + \frac{\partial}{\partial x}\left(k(\xi)\frac{\partial T^{(1)}}{\partial \xi}\right) + \frac{\partial}{\partial x}\left(k(\xi)\frac{\partial T^{(0)}}{\partial x}\right) - \rho(\xi)c(\xi)\frac{\partial T^{(0)}}{\partial t} = 0; \tag{1.9}$$

....

$$\frac{\partial}{\partial \xi}\left(k(\xi)\frac{\partial T^{(n+2)}}{\partial \xi}\right) + \frac{\partial}{\partial \xi}\left(k(\xi)\frac{\partial T^{(n+1)}}{\partial x}\right) + \frac{\partial}{\partial x}\left(k(\xi)\frac{\partial T^{(n+1)}}{\partial \xi}\right) + \frac{\partial}{\partial x}\left(k(\xi)\frac{\partial T^{(n)}}{\partial x}\right) - \rho(\xi)c(\xi)\frac{\partial T^{(n)}}{\partial t} = 0, \tag{1.10}$$

where $\rho(\xi)$, $c(\xi)$, and $k(\xi)$ are periodic functions with a period H.

Substituting the expansion (1.6) into the initial and boundary conditions after equating coefficients of equal powers ε yields

$$T^{(n)}(x,\xi,0) = \begin{cases} T_0, & n = 0 \\ 0, & n \geqslant 1 \end{cases} \quad \text{at} \quad x > 0,$$

$$T^{(n)}(0,\xi,t) = \begin{cases} T_1, & n = 0 \\ 0, & n \geqslant 1 \end{cases} \quad \text{at} \quad t > 0.$$

$$[T^{(n)}(x,\xi,t)]\Big|_G = 0, \qquad n = 0,1,2\ldots,$$

$$\left[k(x)\,\frac{\partial T^{(n)}(x,\xi,t)}{\partial x}\right]\Big|_G = 0, \qquad n = 0,1,2\ldots.$$

Assuming formally x and ξ to be independent variables, we shall consider relation (1.7)–(1.10) as a recurrent chain of equations (differential with respect to ξ) for the unknown functions $T^{(i)}(x,\xi,t)$ $(i = 0,1,2,\ldots,n)$ and appropriate initial and boundary conditions. In this case x is assumed to be a parameter in Eqs. (1.7)–(1.10).

The functions $T^{(i)}(x,\xi,t)$ $(i = 0,1,2,\ldots,n)$ are periodic in ξ. Following [7], integrating Eq. (1.7) with respect to the "fast" variable with subsequent averaging over the period shows that $T^{(0)}(x,\xi,t)$ is not dependent on ξ.

Using the independence condition $T^{(0)}$ on the "fast" variable ξ in integrating Eqs. (1.8) and (1.9), one can show that the first term in the expansion (1.6) is written as

$$T^{(0)}(x,t) = T_1 - (T_1 - T_0)\,\mathrm{erf}\,(x/2(\chi t)^{1/2}).$$

Here $\mathrm{erf}\,(y)$ is the error integral: $\widehat{K} = \langle 1/k(\xi)\rangle^{-1}$; $\chi = \widehat{K}/\langle\rho c\rangle$ is the effective coefficient of thermal conductivity; angle brackets denote averaging over the periodicity cell. The second term of the expansion can be written as

$$T^{(1)}(x,\xi,t) = -(T_1 - T_0)\,\frac{\exp(-x^2/4\chi t)}{(\pi\chi t)^{1/2}} \int_0^{\xi} \left(\frac{\widehat{K}}{k(\eta)} - 1\right) d\eta,$$

and, therefore, after returning to the variables (x,t), the solution of the initial problem in a first approximation will take the form

$$T(x,t) \approx T_1 - (T_1 - T_0)\left\{\mathrm{erf}\left(\frac{x}{2\sqrt{t\chi}}\right) + \frac{\exp(-x^2/4\chi t)}{(\pi\chi t)^{1/2}} \int_0^{x} \left(\frac{\widehat{K}}{k(\eta)} - 1\right) d\eta\right\}. \tag{1.11}$$

As shown in [7], allowing for the zero and the first corrections yields a result close to an exact solution. However, using Eqs. (1.7)–(1.10) one can obtain a recurrent expression for determining the corrections of higher order and a solution of the problem with an arbitrary preassigned accuracy.

2. Now we turn our attention to the case where the temperature T_1 exceeds the melting point of any layer, for instance, the first one $(T_1 > T_1^*)$. Let us consider melting as an instantaneous process. Within this approach different phases are separated by a flat moving surface (melting front). In this case additional boundary conditions other than initial (1.2) and boundary (1.3) conditions should be assigned for the initial problem (1.1) at the melting front. The first of the additional conditions follows from the constancy of temperature at the phase transition boundary

$$T(x,t)\Big|_{x=x_f(t)} = T_1^*, \tag{2.1}$$

and the second is the heat balance equation

$$\left[k(x)\,\frac{\partial T(x,t)}{\partial x}\right]\Big|_{x=x_f(t)} = \rho(x_f)\lambda(x_f)\,\frac{dx_f}{dt}, \tag{2.2}$$

where λ is the specific heat of melting and $x_f(t)$ is the coordinate of the melting front, the law of variation of which is determined in solving the problem.

It follows formally from Eq. (2.1) that the melting front propagates in the media without phase transition. In this case, for such media in Eq. (2.2) one should take $\lambda = 0$, and the equation goes to the appropriate condition (1.5). Following the asymptotic averaging technique, the solution of the problem stated will be sought again as expansion (1.6). Substitution of the expansion in (1.1) gives for the functions $T^{(n)}(x,\xi,t)$ $(n = 0,1,2,\ldots)$ the already known system of equations (1.7)–(1.10). Boundary and initial conditions in a zero approximation have the form

$$T^{(0)}(x,\xi,0)=T_0 \quad \text{at } x>0, \qquad T^{(0)}(0,\xi,t)=T_1 \quad \text{at } t\geqslant 0.$$
$$T^{(0)}(x,\xi,t)\Big|_{\substack{x=x_f(t)\\ \xi=\xi_f(t)}}=T_1^*, \qquad \left[k(\xi)\frac{\partial T^{(0)}}{\partial \xi}\right]\Big|_{\substack{x=x_f(t)\\ \xi=\xi_f(t)}}=0, \tag{2.3}$$

and in a first approximation respectively

$$T^{(1)}(x,\xi,0)=0 \quad \text{at } x>0, \qquad T^{(1)}(0,\xi,t)=0 \quad \text{at } t\geqslant 0;$$
$$T^{(1)}(x,\xi,t)\Big|_{\substack{x=x_f(t)\\ \xi=\xi_f(t)}}=0; \tag{2.4}$$

$$\left[k(\xi)\left(\frac{\partial T^{(0)}}{\partial x}+\frac{\partial T^{(1)}}{\partial \xi}\right)\right]\Big|_{\substack{x=x_f(t)\\ \xi=\xi_f(t)}}=\lambda(\xi_f)\rho(\xi_f)\frac{dx_f}{dt}. \tag{2.5}$$

It should be noted that the transition to the coordinates x and ξ results in the necessity of formal introduction of the law of change of the melting front position in a cell $\xi_f(t)$ in time.

For convenience we introduce the following notation: $\rho_-(\xi)$, $c_-(\xi)$, and $k_-(\xi)$ are the characteristics of the medium behind the melting front, whereas $\rho_+(\xi)$, $c_+(\xi)$, and $k_+(\xi)$ are analogous characteristics ahead of the melting front. From Eq. (1.7) it follows that the zero-order correction can be written in a general form as

$$T^{(0)}(x,\xi,t)=\begin{cases} A(x,t)\int d\xi/k_-(\xi)+v_0(x,t), & 0\leqslant x\leqslant x_f(t),\\ A(x,t)\int d\xi/k_+(\xi)+v_0(x,t), & x>x_f(t). \end{cases} \tag{2.6}$$

It can be shown that the zero-order correction (2.6) will satisfy conditions (2.3) for any position of the front $x_f\geqslant 0$ only when $A(x,t)\equiv 0$, i.e., as in the absence of phase transitions, in the zero approximation the solution of the problem is also independent of the "fast" variable ξ, and

$$T^{(0)}(x,\xi,t)=v_0(x,t). \tag{2.7}$$

In this case the first-order correction is

$$T^{(1)}(x,\xi,t)=\begin{cases} N_1^-(\xi)\dfrac{\partial v_0(x,t)}{\partial x}+C_1(x,t), & 0\leqslant x\ll x_f(t),\\ N_1^+(\xi)\dfrac{\partial v_0(x,t)}{\partial x}+C_2(x,t), & x>x_f(t). \end{cases}$$

where

$$N_1^-(\xi)=\int_0^{\xi}\left(\frac{\langle 1/k_-\rangle^{-1}}{k_-(\eta)}-1\right)d\eta, \qquad N_1^+(\xi)=\int_{\xi}^{H}\left(\frac{\langle 1/k_+\rangle^{-1}}{k_+(\eta)}-1\right)d\eta. \tag{2.8}$$

In Eq. (2.8) when defining the functions $N_1^-(\xi)$ and $N_1^+(\xi)$ we used expression (1.8) and the condition that the medium ahead of the melting front retains periodicity, and in the process of phase transition a periodic medium forms behind the melting front with a periodicity cell characterized already by other properties.

Taking account of Eq. (2.8) and boundary condition (2.5), we obtain an equation for defining the law of motion of the melting front:

$$\langle 1/k_+\rangle^{-1}\frac{\partial v_0}{\partial x}\Big|_{x=x_f+0}-\langle 1/k_-\rangle^{-1}\frac{\partial v_0}{\partial x}\Big|_{x=x_f-0}=\langle\lambda\rho\rangle\frac{dx_f}{dt}. \tag{2.9}$$

Since the function $v_0(x,t)$, being a zero-order correction to the solution, describes an averaged medium, in this equation melting is considered as a continuous process which occurs in a medium with averaged properties.

Substituting (2.7) and (2.8) in Eq. (1.9) after integration with respect to ξ yields:

with $0 \leqslant x \leqslant x_f(t)$

$$k_-(\xi)\frac{\partial T^{(2)}}{\partial \xi} = \int \left(\rho_- c_- \frac{\partial v_0(x,t)}{\partial t} - \langle 1/k_- \rangle^{-1} \frac{\partial^2 v_0(x,t)}{\partial x^2}\right) d\xi - k_- N_1^-(\xi) \frac{\partial^2 v_0(x,t)}{\partial x^2} - k_- \frac{\partial C_1(x,t)}{\partial x} + D(x,t), \tag{2.10}$$

and with $x > x_f(t)$

$$k_+(\xi)\frac{\partial T^{(2)}}{\partial \xi} = \int \left(\rho_+ c_+ \frac{\partial v_0(x,t)}{\partial t} - \langle 1/k_+ \rangle^{-1} \frac{\partial^2 v_0(x,t)}{\partial x^2}\right) d\xi - k_+ N_1^+(\xi) \frac{\partial^2 v_0(x,t)}{\partial x^2} - k_+ \frac{\partial C_2(x,t)}{\partial x} + D(x,t).$$

One can consider the medium in the cells ahead of the melting front to be periodic, which means application of the averaging procedure as was done in deriving Eq. (1.11). A periodical medium is formed behind the melting front. To obtain again the solution (1.11) in the limit in passing to a periodical medium without front, the equality

$$\left\langle k_{+,-}\left(\frac{\partial T^{(2)}}{\partial \xi} + N_1^{+,-}(\xi)\frac{\partial^2 v_0}{\partial x^2} + \frac{\partial C_{1,2}}{\partial x}\right) - D(x,t)\right\rangle = 0 \tag{2.11}$$

should hold, and then averaging of the expressions (2.10) yields

$$\langle \rho_- c_- \rangle \frac{\partial v_0(x,t)}{\partial t} - \langle 1/k_- \rangle^{-1} \frac{\partial^2 v_0(x,t)}{\partial x^2} = 0 \qquad \text{at } 0 \leqslant x \leqslant x_f(t),$$
$$\langle \rho_+ c_+ \rangle \frac{\partial v_0(x,t)}{\partial t} - \langle 1/k_+ \rangle^{-1} \frac{\partial^2 v_0(x,t)}{\partial x^2} = 0 \qquad \text{at } x > x_f(t).$$

Equality (2.11) is an additional condition for determining the second-order correction $T^{(2)}$ ahead and behind the melting front. Denoting

$$\chi_1 = \frac{\langle 1/k_- \rangle^{-1}}{\langle \rho_- c_- \rangle}, \qquad \chi_2 = \frac{\langle 1/k_+ \rangle^{-1}}{\langle \rho_+ c_+ \rangle},$$

for determining $v_0(x,t)$ we have the problem

$$\begin{aligned} \chi_1 \frac{\partial^2 v_0(x,t)}{\partial x^2} &= \frac{\partial v_0(x,t)}{\partial t} \qquad \text{at} \quad 0 \leqslant x \leqslant x_f(t), \\ \chi_2 \frac{\partial^2 v_0(x,t)}{\partial x^2} &= \frac{\partial v_0(x,t)}{\partial t} \qquad \text{at} \quad x > x_f(t) \end{aligned} \tag{2.12}$$

with boundary condition (2.9) and the conditions

$$v_0(x,0) = T_0, \; x > 0, \quad v_0(0,t) = T_1, \; t \geqslant 0, \quad v_0(x,t)|_{x=x_f(t)} = 0, \tag{2.13}$$

obtained by substituting (2.7) into the first three expressions of (2.3).

One can easily see that problem (2.12) with conditions (2.9) and (2.13) is an analog of the known Stefan problem (see [1]). Therefore, its solutions should have the form

$$v_0(x,t) = \begin{cases} T_1 + (T_1^* - T_1)\dfrac{\operatorname{erf}(x/2(\chi_1 t)^{1/2})}{\operatorname{erf}(\alpha/2(\chi_1)^{1/2})}, & 0 \leqslant x \leqslant x_f(t), \\ T_0 + (T_1^* - T_0)\dfrac{\operatorname{erf}(x/2(\chi_2 t)^{1/2}) - 1}{\operatorname{erf}(\alpha/2(\chi_2)^{1/2}) - 1}, & x > x_f(t) \end{cases} \tag{2.14}$$

(α is a constant connected with the front position $x_f(t)$ by the relation $x_f(t) = \alpha(t)^{1/2}$).

Boundary condition (2.9) with account of (2.14) gives the following equation for determining the constant α:

$$(\chi_2)^{1/2}\langle \rho_+ c_+ \rangle (T_1^* - T_0)\frac{\exp(-\alpha^2/4\chi_2)}{\operatorname{erf}(\alpha/2(\chi_2)^{1/2}) - 1} - (\chi_1)^{1/2}\langle \rho_- c_- \rangle (T_1^* - T_1)\frac{\exp(-\alpha^2/4\chi_1)}{\operatorname{erf}(\alpha/2(\chi_1)^{1/2})} = \frac{(\pi)^{1/2}}{2}\alpha\langle \lambda\rho \rangle, \tag{2.15}$$

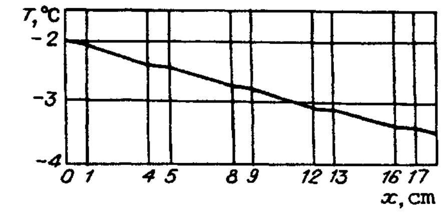

Fig. 2

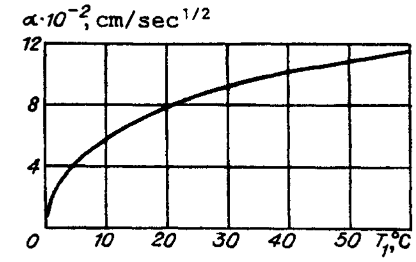

Fig. 3

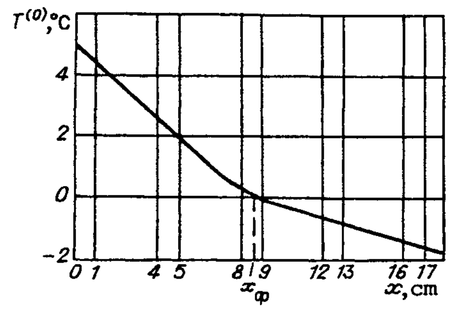

Fig. 4

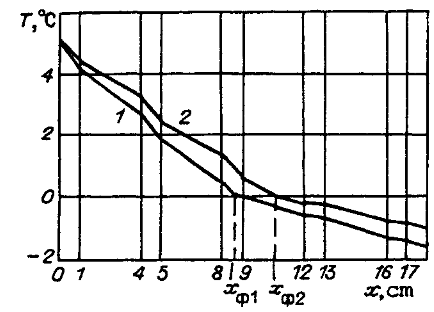

Fig. 5

which can be solved numerically.

To obtain the first-order correction $T^{(1)}(x,\xi,t)$, we substitute (2.14) into Eq. (2.8) and using condition (2.4) for the case of approaching the front from the left we obtain

$$T^{(1)}\Big|_{\substack{x=x_f-0\\ \xi=\xi_f-0}}=\int\limits_0^{\xi_f}\left(\frac{\langle 1/k_-\rangle^{-1}}{k_-(\eta)}-1\right)d\eta\frac{\exp(-x_f^2/4\chi_1 t)}{\operatorname{erf}(\alpha/2\sqrt{\chi_1})}\frac{T_1^*-T_1}{\sqrt{\chi_1\pi t}}+C_1(x_f-0,t)=0$$

and similarly when approaching from the right

$$T^{(1)}\Big|_{\substack{x=x_f+0\\ \xi=\xi_f+0}}=\int\limits_{\xi_f}^{H}\left(\frac{\langle 1/k_+\rangle^{-1}}{k_+(\eta)}-1\right)d\eta\frac{\exp(-x_f^2/4\chi_2 t)}{\operatorname{erf}(\alpha/2\sqrt{\chi_2})-1}\frac{T_1^*-T_0}{\sqrt{\chi_2\pi t}}+C_2(x_f+0,t)=0.$$

From the last two equations one can easily express C_1 and C_2 at the point where the front is. But since $x_f(t)$ can take up all possible x, the expressions obtained for $C_1(x_f,t)$ and $C_2(x_f,t)$ should be valid for all x:

$$C_1(x,t)=\int\limits_0^{\xi_f}\left(1-\frac{\langle 1/k_-\rangle^{-1}}{k_-(\eta)}\right)d\eta\,\frac{\exp(-x^2/4\chi_1 t)}{\operatorname{erf}(\alpha/2(\chi_1)^{1/2})}\,\frac{T_1^*-T_1}{(\chi_1\pi t)^{1/2}},$$

$$C_2(x,t)=\int\limits_{\xi_f}^{H}\left(1-\frac{\langle 1/k_+\rangle^{-1}}{k_+(\eta)}\right)d\eta\,\frac{\exp(-x^2/4\chi_2 t)}{\operatorname{erf}(\alpha/2(\chi_2)^{1/2})-1}\,\frac{T_1^*-T_0}{(\chi_2\pi t)^{1/2}}.$$

Thus, the first-order correction $T^{(1)}(x,\xi,t)$ is determined completely and one can write the solution o problem (1.1) with conditions (1.2), (1.3) and (2.1), (2.2), which in the first approximation is of the form

$$T(x,t) = \begin{cases} T_1 + \dfrac{T_1^* - T_1}{\operatorname{erf}(\alpha/2(\chi_1)^{1/2})}\left\{\operatorname{erf}\left(\dfrac{x}{2(\chi_1 t)^{1/2}}\right)\right. \\ \left. \quad + \dfrac{\exp(-x^2/4\chi_1 t)}{(\chi_1 \pi t)^{1/2}} \displaystyle\int_x^{x_f(t)} \left(1 - \dfrac{\langle 1/k_- \rangle^{-1}}{k_-(\eta)}\right) d\eta \right\}, & 0 \leqslant x \leqslant x_f(t), \\ T_0 + \dfrac{T_1^* - T_0}{\operatorname{erf}(\alpha/2(\chi_2)^{1/2}) - 1}\left\{\operatorname{erf}\left(\dfrac{x}{2(\chi_2 t)^{1/2}}\right) - 1\right. \\ \left. \quad + \dfrac{\exp(-x^2/4\chi_2 t)}{(\chi_2 \pi t)^{1/2}} \displaystyle\int_{x_f(t)}^{x} \left(1 - \dfrac{\langle 1/k_+ \rangle^{-1}}{k_+(\eta)}\right) d\eta \right\}, & x > x_f(t). \end{cases} \tag{2.16}$$

Expression (2.16) in the first approximation describes the distribution of temperature in a layered medium with allowance for the phase transition processes.

Figure 2 shows the results of calculating the temperature distribution in the first periodicity cells 12 hours after the moment when the heating of the medium with initial temperature $T_0 = -5°C$ began. In this case the temperature $T_1 = -2°C$ was kept at the free surface. The calculation was carried out up to the second-order correction in expansion (1.6), which made it possible to take account of the effect of the medium inhomogeneity. It should be noted that within each layer the temperature variation follows a law close to a linear one. This is in good agreement with the numerical results.

With the proviso that $T_1 > 0$, melting processes start in the medium. In the context of the given model the motion of the melting front is determined by the parameter α. Figure 3 shows the dependence of the parameter on the temperature at the free surface. In calculations the water resulting from the ice melting was characterized by the parameters $\rho_3 = 1{,}000$ kg/m^3, $c_3 = 4{,}200$ J/(kg · deg), $k_3 = 0.567$ W/(m · deg). Figures 4 and 5 present the results of calculations of the temperature distribution at $T_1 = 5°C$ and $T_0 = -5°C$ with the ice–water phase transitions in the medium. Figure 4 presents the value of zero correction $T^{(0)}$ for the time moment $t = 12$ hours when the melting front is at a distance $x_f = 8.54$ cm from the free surface. The dependence $T^{(0)}$ on x is of smooth character, and even the position of the phase transition front is not a breakdown point. However, account of the first correction (Fig. 5, curve 1) results in the appearance of the breakdown points at the layer boundaries and in the melting front. Curve 2 shows the calculation results for the time $t = 18$ hours when the melting front interpreted using this approach as a surface with temperature $T_1^* = 0$ is in the clay. A comparison shows good agreement between the calculation results for the dynamics of temperature variation in a periodic medium obtained with the use of the asymptotic averaging technique and numerical methods.

Thus, the approach developed within the framework of the asymptotic averaging technique makes it possible to describe the heat propagation process in periodic media, including allowance for the phase transitions, by means of analytical relations. A substantial advantage of the method is that it enables one to take account of a great number of layers within one periodicity cell with arbitrary dimensions and physical characteristics.

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